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Initial-Value Problem for Artificial Satellites Motion in The Earth's Gravitational Field with Axial Symmetry Using Eulerian Parameters

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ABSTRACT. In this paper, a general recursive and stable computational algorithm is established for the initial-value problem of the Eulerian parameters for satellites prediction in the Earth's gravitational field with axial symmetry. Applications of the algorithm are also given.

Introduction

The usage of either the analytical or numerical techniques on the conventional equations of space dynamics for the motion of space vehicles and artificial satellites yield inaccurate prediction for their positions and velocities. This is because that, these equations are nearly singular for the cases of close approach, which are of common occurrence in mission and re-entry problems of the space travel. Such events produce large gravitational forces and sharp bend which in turn causes poor prediction. Due to the great importance of getting very accurate predictions of these bodies for the practical and military purposes, which in turn has led several authors to establish new forms of the equations of motion capable of studying the perturbed orbits of these bodies. The author's ideas relied on transforming the equations of motion to a form which is characterized by very stable properties with respect to the numerical as well as the analytical integrations.

One such transformation, is the Eulerian parameters[1]. In the present paper, a general recursive and stable computational algorithm is established for the initial-value problem of the Eulerian parameters for satellites prediction in the Earth's gravitational field with axial symmetry.

Euler Parameter Regularization

The most interesting connection between orbit dynamics and rigid body dynamics is established if one conceives of the orbit normal $\hat{z}$, the radius vector $\hat{x}$, and the orthogonal vector $\hat{y} = \hat{z} \times \hat{x}$ as a rigid body[2]. It is immediately possible to introduce many results from rigid body dynamics into orbital/trajectory dynamics. While this connection may seem a bit artificial, the consequences are in fact significant; the most attractive system of linear differential equations results from this line of reasoning.

Euler Parameters

Let the unit vectors of the inertial axes be $(\hat{x}, \hat{y}, \hat{z})$, then the triad $(\hat{x}, \hat{y}, \hat{z})$ at any instant projects onto $(\hat{i}, \hat{j}, \hat{k})$ according to

$$\hat{x} = \hat{i}\, c_{11} + \hat{j}\, c_{12} + \hat{k}\, c_{13}\,, \tag{2.1.1}$$

$$\hat{y} = \hat{i}\, c_{21} + \hat{j}\, c_{22} + \hat{k}\, c_{23}\,, \tag{2.1.2}$$

$$\hat{z} = \hat{i}\, c_{31} + \hat{j}\, c_{32} + \hat{k}\, c_{33}\,. \tag{2.1.3}$$

The direction cosines (c_{ij}) are usually expressed as functions of three Euler angles; this leads to a system of nonlinear transcendental differential equations which invariably contain a singularity. Alternatively, one can introduce four Euler parameters and avoid these problems altogether. Although the $(\hat{x}, \hat{y}, \hat{z})$ axes are nearly spinning about $\hat{z}$ with angular velocity $d\varphi / dt$, where φ is the true anomaly (a part of a constant), the perturbations cause departures from this Keplerian truth which must be accounted for. As a function of Euler parameters, the direction cosines are

$$c_{11} = u_1^2 - u_2^2 - u_3^2 + u_4^2\,, \tag{2.2.1}$$

$$c_{12} = 2\,(u_1\, u_2 + u_3\, u_4)\,, \tag{2.2.2}$$

$$c_{13} = 2\,(u_1\, u_3 - u_2\, u_4)\,, \tag{2.2.3}$$

$$c_{21} = 2\,(u_1\, u_2 - u_3\, u_4)\,, \tag{2.2.4}$$

$$c_{22} = -u_1^2 + u_2^2 - u_3^2 + u_4^2\,, \tag{2.2.5}$$

$$c_{23} = 2\,(u_1\, u_4 + u_2\, u_3)\,, \tag{2.2.6}$$

$$c_{31} = 2\,(u_1\, u_3 + u_2\, u_4)\,, \tag{2.2.7}$$

$$c_{32} = 2\,(u_2\, u_3 - u_1\, u_4)\,, \tag{2.2.8}$$

$$c_{33} = -u_1^2 - u_2^2 + u_3^2 + u_4^2\,. \tag{2.2.9}$$

Geometrically, u_i ; i = 1, 2, 3, 4 are closely connected to Euler's *Principal Rotation Theorem*[3], which states that a general rotational transformation can be accomplished by a single rigid rotation (through a principal angle ψ) about a principal unit vector

$$\hat{\ell} = \ell_1 \hat{x} + \ell_2 \hat{y} + \ell_3 \hat{z} . \qquad (2.3)$$

We define

$$u_i = \ell_1 \sin\frac{1}{2}\psi , \qquad \text{for } i = 1,2,3. , \qquad (2.4.1)$$

$$u_4 = \cos\frac{1}{2}\psi . \qquad (2.4.2)$$

Thus

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 \qquad (2.5)$$

The most important kinematical truth is that the Euler parameters rigorously satisfy the bilinear orthogonal relationships[2]

$$\dot{u}_1 = \frac{1}{2}(\omega_1 u_4 - \omega_2 u_3 + \omega_3 u_1) , \qquad (2.6.1)$$

$$\dot{u}_2 = \frac{1}{2}(\omega_1 u_3 + \omega_2 u_4 - \omega_3 u_1) , \qquad (2.6.2)$$

$$\dot{u}_3 = \frac{1}{2}(-\omega_1 u_2 + \omega_2 u_1 + \omega_3 u_4) , \qquad (2.6.3)$$

$$\dot{u}_4 = \frac{1}{2}(-\omega_1 u_1 - \omega_2 u_2 - \omega_3 u_3) , \qquad (2.6.4)$$

These equations can be written in a matrix form as

$$\dot{u} = \frac{1}{2} B(\omega) u , \qquad (2.7)$$

where

$$B(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \qquad (2.8)$$

Note that the matrix B is skew symmetric, and

$$\omega = \omega_1 \hat{x} + \omega_2 \hat{y} + \omega_3 \hat{z} , \qquad (2.9)$$

is the instantaneous angular velocity of $(\hat{x}, \hat{y}, \hat{z})$. The angular velocity of $(\hat{x}, \hat{y}, \hat{z})$, for Keplerian motion is

$$\omega_1 = \omega_2 = 0 \,, \tag{2.10.1}$$

$$\omega_3 = \frac{d\varphi}{dt} = \frac{\sqrt{\mu p^*}}{r^2} \,, \tag{2.10.2}$$

where p* is the semi-latus rectum of osculating orbit, μ is the Earth's gravitational constant and r is the magnitude of the radius vector **r**.

Change of Variables

Equations (2.6) can be modified significantly by choosing a variable other than time as independent variable. Vitins[2] pursued these ideas with the following variable changes:

$$\hat{x} = \frac{\mathbf{r}}{r} = (u_1^2 - u_2^2 - u_3^2 + u_4^2)\hat{i} + 2(u_1u_2 + u_3u_4)\hat{j} + 2(u_1u_3 - u_3u_4)\hat{k}, \tag{2.11.1}$$

$$\rho = \frac{1}{r} \,, \tag{2.11.2}$$

$$p = p^* + \frac{2r^2}{\mu} V \,, \tag{2.11.3}$$

$$dt = \frac{r^2 \nu}{\sqrt{\mu p^*}} d\varphi = \frac{1}{\sqrt{\mu p}} \cdot \frac{1}{\rho^2} d\varphi \,, \tag{2.11.4}$$

$$\nu = \sqrt{\frac{p^*}{p}} = \sqrt{1 - \frac{2r^2}{\mu p} V} \tag{2.11.5}$$

where V is the perturbed time-independent potential.

Satellite Motions in Terms of Euler Parameters

The differential equation in rectangular variables r = (x, y, z) describing the rate of change of the position and velocity for a satellite orbiting the Earth is given by

$$\ddot{r} + \mu \frac{\mathbf{r}}{r^3} = p = p^* - \frac{\partial V}{\partial r} \,, \tag{2.12}$$

where p* is the resultant of all non-conservative perturbing forces and forces derivable from a time-dependent potential. Upon introducing the transformations of Equations (2.11), we get the following equations:

$$\frac{du_1}{d\varphi} = \frac{1}{2}(\omega_1 u_4 + \omega_3 u_2) \ , \tag{2.13.1}$$

$$\frac{du_2}{d\varphi} = \frac{1}{2}(\omega_1 u_3 - \omega_3 u_1) \ , \tag{2.13.2}$$

$$\frac{du_3}{d\varphi} = \frac{1}{2}(-\omega_1 u_2 + \omega_3 u_4) \ , \tag{2.13.3}$$

$$\frac{du_4}{d\varphi} = \frac{1}{2}(-\omega_1 u_1 - \omega_3 u_3) \ , \tag{2.13.4}$$

$$\frac{d^2\rho}{d\varphi^2} + \rho = \frac{1}{p} g_1 \ , \tag{2.13.5}$$

$$\frac{dp}{d\rho} = g_2 \ , \tag{2.13.6}$$

$$\frac{dt}{d\varphi} = \frac{1}{\sqrt{\mu p}} \cdot \frac{1}{\rho^2} \cdot \tag{2.13.7}$$

where

$$g_1 = \frac{1}{\mu p \rho^2} [< p, \hat{x} > -2\rho V] - \frac{1}{2\rho} \frac{d\rho}{d\varphi} g_2 \tag{2.14.1}$$

$$g_2 = \frac{2\nu}{\mu \rho^3} < p^*, \hat{y} > + \frac{2}{\mu \sqrt{\mu p}} \frac{1}{\rho^4} \frac{\partial V}{\partial t} - \frac{2}{\mu \rho^3} \left[2V + < r, \frac{\partial V}{\partial r} > \right] \frac{d\rho}{d\phi} \tag{2.14.2}$$

$$\omega_1 = \frac{1}{\mu p \rho^3 \upsilon} < p, \hat{z} > \quad ; \quad \omega_3 = \upsilon = \sqrt{1 - \frac{2r^2}{\mu p} V} \tag{2.14.3}$$

$< a, b >$ is used to denote the scalar product of the two vectors a and b.

Keplerian Motion

The general differential Equations (2.13) for Keplerian motion ($p = V = p^* = 0$) reduce to

$$\frac{du_1}{d\varphi} = \frac{1}{2} u_2 \ , \tag{2.15.1}$$

$$\frac{du_2}{d\varphi} = -\frac{1}{2} u_1 \ , \tag{2.15.2}$$

$$\frac{du_3}{d\varphi} = \frac{1}{2} u_4 \ , \tag{2.15.3}$$

$$\frac{du_4}{d\varphi} = -\frac{1}{2} u_3 \ , \tag{2.15.4}$$

$$\frac{d^2\rho}{d\varphi^2} = -\rho + \frac{1}{p} \ , \tag{2.15.5}$$

$$\frac{dp}{d\varphi} = 0 \rightarrow p = \text{constant} = p^* \ , \tag{2.15.6}$$

$$\frac{dt}{d\varphi} = \frac{1}{\sqrt{\mu p}} \frac{1}{\rho^2} \tag{2.15.7}$$

where φ is the change in true anomaly. Integrating the above Equations we get for Keplerian solution:

$$u_1 = \alpha_1 \cos \frac{1}{2}\varphi + \alpha_2 \ \sin \frac{1}{2} \ \varphi \ , \tag{2.16.1}$$

$$u_2 = -\alpha_1 \sin \frac{1}{2}\varphi + \alpha_2 \ \cos \frac{1}{2} \ \varphi \ , \tag{2.16.2}$$

$$u_3 = \alpha_3 \cos \frac{1}{2}\varphi + \alpha_4 \ \sin \frac{1}{2} \ \varphi \ , \tag{2.16.3}$$

$$u_4 = -\alpha_3 \sin \frac{1}{2}\varphi + \alpha_5 \cos \frac{1}{2} \ \varphi \ , \tag{2.16.4}$$

$$\rho = \frac{1}{p^*} + \alpha_5 \cos\varphi + \alpha_6 \sin\varphi \ , \tag{2.16.5}$$

$$\frac{d\rho}{d\varphi} = -\alpha_5 \sin\varphi + \alpha_6 \cos\varphi \ , \tag{2.16.6}$$

$$\alpha_7 = p^* \ , \tag{2.16.7}$$

$$t = \alpha_8 + F(\varphi) = \text{classical Kepler's equation.} \tag{2.16.8}$$

The α's are constants evaluated from initial conditions.

Satellite Motion in the Earth's Gravitational Field with Axial Symmetry

In this section, the initial value problem of the Eulerian parameters will be considered in detail for satellite motion in the Earth's gravitational field with axial symmetry. A general, recursive and stable computational algorithm of this problem will be developed for any conic motion and for any number N ≤ 2 of the zonal harmonic coefficients of the Earth's gravitational potential. Applications of the algorithm are given in section 3.2.

Expressions of V, ∂V / ∂r, and ($p_{\hat{x}}, p_{\hat{y}}, p_{\hat{z}}$)

For the case of axial symmetry we have

$$P^* = 0 \ , \ \frac{\partial V}{\partial t} = 0 \ , \tag{3.1}$$

$$V = \left(\frac{\mu}{R}\right)\sum_{k=2}^{N} J_k \left(\frac{R}{r}\right)^{k+1} P_k(\sin\Gamma) \ , \tag{3.2}$$

where V is the Earth's gravitational field with axial symmetry, R is the Earth's mean equatorial radius; Γ the latitude of the satellite, J_k , k = 2, 3, ... , N are dimensionless numerical coefficients (note that the infinite series of Equation (3.2) is truncated at some positive integer N), and $P_\ell(Z)$ is the Lagendre polynomial in Z of order ℓ with $Z \in [-1, 1]$. By the same argument as in[4], it could be shown that the economical and stable recurrent computations of V and ∂V / ∂r in terms of the Eulerian parameters are

$$V = \left(\frac{\mu}{R}\right)\sum\nolimits_{k=2}^{N} J_k Q_k \ , \tag{3.3}$$

$$\frac{\partial V}{\partial x} = -\left(\frac{\mu c_{12}\rho^2}{1-c_{13}^2}\right) G \ , \tag{3.4}$$

$$\frac{\partial V}{\partial y} = -\left(\frac{\mu c_{12}\rho^2}{1-c_{13}^2}\right) G \ , \tag{3.5}$$

$$\frac{\partial V}{\partial z} = -\mu\rho^2 S \ , \tag{3.6}$$

where

$$\rho = \frac{1}{r}; G = \sum_{K=2}^{N} I_K F_K \ ; \ S = \sum_{K=2}^{N} I_K D_K \ ; \ I_K = (k+1)J_K \ , \tag{3.7}$$

Q's and D's satisfy economical and stable recurrence formulae of the forms :

$$Q_K = Q_o[c_{13}Q_{K-1} - Q_o Q_{K-2} + c_{13}Q_{K-1} - (c_{13}Q_{K-1} - Q_o Q_{K-2})/k] \ , \tag{3.8}$$

$$D_K = Q_o[D_o D_{K-1} - Q_o D_{K-2} + D_o D_{K-1} - (D_o D_{K-1} - Q_o D_{K-1})/(k+1)] \;, \quad (3.9)$$

$$F_K = Q_o D_{K-1} - D_o D_K \;, \quad (3.10)$$

$$Q_o = R\rho \;;\; Q_1 = Q_0^2 c_{13} \;;\; D_o = c_{13} \;,\; D_1 = 0.5\; Q_o(3\; D_o^2 - 1) \;, \quad (3.11)$$

c's are given in terms of u's by Equations (2.2).

Finally, by using Equations (3.4), (3.5) and (3.6) we get

$$P_{\hat{x}} = \langle P, \hat{x} \rangle = \left\langle -\frac{\partial V}{\partial r}, \hat{x} \right\rangle = \mu\rho^2 Q_o \sum_{K=2}^{N} I_K D_{K-1} \;, \quad (3.12)$$

$$P_{\hat{y}} = \langle P, \hat{y} \rangle = \left\langle -\frac{\partial V}{\partial r}, \hat{y} \right\rangle = \frac{\mu c_{23} \rho^2}{1 - c_{13}^2} \sum_{K=2}^{N} I_K (D_K - D_o Q_o D_{K-1}) \;, \quad (3.13)$$

$$P_{\hat{z}} = \langle P, \hat{z} \rangle = \left\langle -\frac{\partial V}{\partial r}, \hat{z} \right\rangle = \frac{\mu c_{33} \rho^2}{1 - c_{13}^2} \sum_{K=2}^{N} I_K (D_K - D_o Q_o D_{K-1}) \;. \quad (3.14)$$

In what follows, some fundamental computational algorithms for the present initial value problem are given. Each algorithm is described by its purpose, input, and its computational sequence.

Computational Algorithms

Computational Algorithm (1)

Purpose: To compute u_j ; j = 1,2,...,8 from r, $\dot{r}$ and V(r) at any epoch to, where (u_1, u_2, u_3, u_4) are the Euler parameters, $u_5 = \rho = \frac{1}{r}, u_6 = \frac{d\rho}{d\varphi} = \rho', u_7 = p$, $u_8 = t_o$. [Note, for simplicity of notation, we shall use hereafter (x_1, x_2, x_3) to refer respectively to (x, y, z)].

Input: t_o ; μ ; r ; $\dot{r}$; V(r)

Computational Sequence:

(1) $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

(2) $\dot{r} = (x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3)/r$

(3) $u_5 = 1/r$

(4) $u_7 = \frac{r^2}{\mu}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - \dot{r}^2 + 2V)$

(5) $u_6 = -\dot{r}(\mu u_7)^{-1/2}$

(6) $c_{11} = x_1 / r$

(7) $c_{12} = x_2 / r$

(8) $c_{13} = x_3 / r$

(9) $\nu = \left(1 - \frac{2r^2}{\mu u_7} V\right)^{1/2}$

(10) $S = \nu(\mu u_7)^{1/2}$

(11) $c_{21} = (r\dot{x}_1 - \dot{r}x_1) / S$

(12) $c_{22} = (r\dot{x}_2 - \dot{r}x_2) / S$

(13) $c_{23} = (r\dot{x}_3 - \dot{r}x_3) / S$

(14) $c_{31} = c_{12}c_{23} - c_{13}c_{22}$

(15) $c_{32} = c_{13}c_{21} - c_{11}c_{23}$

(16) $c_{33} = c_{11}c_{22} - c_{12}c_{21}$

(17) $u_4 = (1 + c_{11} + c_{22} + c_{33})^{1/2} / 2$

(18) if $u_4 = 0$ go to 23 (note that, $u_4 = 0$ means $\psi = 180^o$ and this has no meaning)

(19) $u_i = 0.25(c_{23} - c_{32}) / u_4$

(20) $u_2 = 0.25(c_{31} - c_{13}) / u_4$

(21) $u_3 = 0.25(c_{12} - c_{21}) / u_4$

(22) $u_8 = t_o$

(23) END

Computational Algorithm (2)

Purpose: To compute V using the recursive formulations of Section 3.1 in terms of:

(a) The position vector **r** or

(b) Euler parameters u_j and $u_5 = \rho = \frac{1}{r}; j = 1, 2, 3, 4.$

Input: μ, R, N, J_k ; k = 1, 2,...,N, IT

Where IT = 1, if it is required to compute V from r

IT = 2, if it is required to compute V from u_j.

T is a given vector of dimension five such that its components are:

$(x_1, x_2, x_3, 0, 0)$ if IT = 1 or

$(u_1, u_2, u_3, u_4, u_5)$ if IT = 2.

Computational Sequences

(1) If IT = 2 go to step 6

(2) $r = (T_1^2 + T_2^2 + T_3^2)^{1/2}$

(3) $Q_o = R / r$

(4) $E = T_3 / r$

(5) Go to step 8

(6) $Q_o = R\ T_5$

(7) $E = 2(T_1T_3 - T_2T_4)$

(8) $H_1 = Q_o^2\ E$

(9) $S \leftarrow 0$

(10) $H_o \leftarrow Q_o$

(11) $H_2 \leftarrow H_1$

(12) For all k = 2(1)N Compute

$A = Q_oH_0$; $G = E\ H_2$; $B = G - A$;

$H_2 = Q_o\ (B + G - B / k)$;

$H_o \leftarrow H_1$; $H_1 \leftarrow H_2$; $S \leftarrow S + J_K\ H_2$.

(13) $V = (\mu / R)\ S$

(14) END.

Note that, the case in which IT = 1 of this algorithm could be used in computing the total energy.

Computational Algorithm (3)

Purpose: To compute $\partial V / \partial r$ using the recursive formulations of Section 3.1 in terms of:

(a) The position vector r
(b) Euler parameters u_j and $u_5 = \rho = \frac{1}{r}; j = 1, 2, 3, 4.$

Input: μ, R, N, J_K ; k = 1, 2,...,N , IT

where

IT = 1 if it is required to compute $\partial V / \partial r$ from r
IT = 2 if it is required to compute $\partial V / \partial r$ from u_j ; j = 1,2,3,4,5.
T is a given vector of dimension five such that its components are :

$(x_1, x_2, x_3, 0, 0)$ if IT = 1
$(u_1, u_2, u_3, u_4, u_5)$ if IT = 2.

Computational Sequence

(1) if IT = 2 Go to step 7

(2) $r_1^2 = T_1^2 + T_2^2$

(3) $r^2 = r_1^2 + T_3^2$

(4) $D_o = T_3 / r$

(5) $Q_o = R / r$

(6) Go to step 11

(7) $Q_o = R\, T_5$

(8) $D_o = 2\,(T_1 T_3 - T_2 T_4)$

(9) $c_{11} = T_1^2 - T_2^2 - T_3^2 + T_4^2$

(10) $c_{12} = 2\,(T_1 T_2 + T_3 T_4)$

(11) $S \leftarrow 0$

(12) $C \leftarrow 0$

(13) $H_o \leftarrow D_o$

(14) $H_1 \leftarrow 0.5\ Q_o\ (3\ D_o^2 - 1)$

(15) $H_2 \leftarrow H_1$

(16) For all k = 2(1) N, compute

$k_1 = k + 1$; $B = H_oQ_o$; $G = D_oH_2$; $A = G - B$;

$H_2 = Q_o\ (A + G - A / k_1)$; $F = Q_oH_1 - D_oH_2$;

$H_o \leftarrow H_1$; $H_1 \leftarrow H_2$; $C \leftarrow C + k_1J_KF$; $S \leftarrow S + k_1J_KH_2$.

(17) If IT = 2 Go to step 22

(18) $\frac{\partial V}{\partial x_1} = -\left(\frac{\mu T_1}{r\, r_1^2}\right)C$

(19) $\frac{\partial V}{\partial x_2} = -\left(\frac{\mu T_2}{r\, r_1^2}\right)C$

(20) $\frac{\partial V}{\partial x_3} = -\left(\frac{\mu}{r^2}\right)C$

(21) Go to step 25

(22) $\frac{\partial V}{\partial x_1} = -\left(\frac{\mu c_{11} T_5^2}{1 - D_o^2}\right)C$

(23) $\frac{\partial V}{\partial x_2} = -\left(\frac{\mu c_{12} T_5^2}{1 - D_o^2}\right)C$

(24) $\frac{\partial V}{\partial x_3} = -\mu T_3^2 S$

(25) END

Computational Algorithm (4)

Purpose: To compute r and $\dot{r}$ from u_j ; j = 1,2,...,7 and

$$u_i' = \frac{du_i}{d\varphi}; i = 1, 2, 3, 4.$$

Input: μ, u_j and u_i' ; j = 1(1)7 ; i = 1(1)4

Computational Sequence

(1) $r = 1/u_5$

(2) $x_1 = r\,(u_1^2 - u_2^2 - u_3^2 + u_4^2)$

(3) $x_2 = 2\,r\,(u_1u_2 + u_3u_4)$

(4) $x_3 = 2\,r\,(u_1u_3 - u_2u_4)$

(5) $S = \dfrac{\sqrt{\mu\,u_7}}{r^2}$

(6) $\dot{x}_1 = \left\{2\,(u_1u_1' - u_2u_2' - u_3u_3' + u_4u_4')\,r - \dfrac{x_1}{u_5}u_6\right\}S$

(7) $\dot{x}_2 = \left\{2\,(u_2u_1' + u_1u_2' + u_4u_3' + u_3u_4')\,r - \dfrac{x_2}{u_5}u_6\right\}S$

(8) $\dot{x}_3 = \left\{2\,(u_3u_1' + u_1u_3' - u_4u_2' - u_2u_4')\,r - \dfrac{x_3}{u_5}u_6\right\}S$

(9) END

Numerical Applications

Test Cases

For the purposes of numerical applications of our algorithm, we consider three fractional orbit cases all with the same initial time $t_o = 0$ sec., while the initial values are listed for each case in the first column of Tables 1, 2 and 3 together with the type of the orbit.

The Adopted Physical Constants

$$\mu = 398600.8\ \text{km}^3\ \text{sec}^{-2} \quad ; \quad R = 6378.135\ \text{km}$$

The numerical values of the Earth's zonal harmonic coefficients J_K ; k = 2,3,...,36 are taken from reference[5].

Checks During Numerical Integration

The accuracy of the computed values during the numerical integration could be checked by the conditions:

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 \,, \tag{3.14}$$

$$\omega_2 = u_2 u_4' + u_3 u_1' - u_4 u_2' - u_1 u_3' = 0. \tag{3.15}$$

TABLE 1. Initial and final states with harmonics up to J_{36} for the test case No. 1 (Units: t_o and TF are in sec., X in km and $\dot{X}$ in km / sec).

Initial	Final	Accuracy checks
Elliptic orbit	Elliptic orbit	
t_o = + 0.0000000000 D + 00	TF = + 0.1800000900 D + 04	Check 1 = 0.1000000000 D + 01
X_1 = + 0.6478000000 D + 04	X_1 = + 0.1097091933 D + 05	Check 2 = 0.9859992902 D – 21
X_2 = + 0.0000000000 D + 00	X_2 = + 0.1435479581 D + 04	Check 3 = 0.2629760517 D – 11
X_3 = + 0.0000000000 D + 00	X_3 = + 0.4304935816 D + 04	
$\dot{X}_1$ = + 0.7000000000 D + 01	$\dot{X}_1$ = – 0.4446857760 D + 00	
$\dot{X}_2$ = + 0.1000000000 D + 01	$\dot{X}_2$ = + 0.5322856247 D + 00	
$\dot{X}_3$ = + 0.3000000000 D + 01	$\dot{X}_3$ = + 0.1595330525 D + 01	

TABLE 2. Initial and final states with harmonics up to J_{36} for the test case No. 2 (Units: t_o and TF are in sec., X in km and $\dot{X}$ in km / sec).

Initial	Final	Accuracy checks
Parabolic orbit	Parabolic orbit	
t_o = + 0.0000000000 D + 00	TF = + 0.1946250000 D + 04	Check 1 = 0.1000000000 D + 01
X_1 = – 0.9592151798 D + 04	X_1 = – 0.1835330831 D + 05	Check 2 = 0.2394687873 D – 20
X_2 = + 0.4539210547 D + 04	X_2 = + 0.1095613853 D + 05	Check 3 = – 0.304914110 D – 08
X_3 = – 0.2198098325 D + 04	X_3 = + 0.6018407546 D + 04	
$\dot{X}_1$ = – 0.6217477283 D + 01	$\dot{X}_1$ = – 0.3465737904 D + 01	
$\dot{X}_2$ = + 0.4184991210 D + 01	$\dot{X}_2$ = + 0.2718405654 D + 01	
$\dot{X}_3$ = + 0.4170160618 D + 01	$\dot{X}_3$ = + 0.4060604939 D + 01	

In addition to these two general conditions, the present problem provides a third one which is the constancy of the total energy (since the potential with axial symmetry is conservative), that is

$$\Delta h = h(t) - h(0) = 0 \tag{3.16}$$

where h(t) and h(0) are the values of the total energy h at any time t and at the initial epoch t = 0, respectively.

TABLE 3. Initial and final states with harmonics up to J_{36} for the test case No. 3 (Units: t_o and TF are in sec., X in km and $\dot{X}$ in km / sec).

Initial	Final	Accuracy checks
Hyperbolic orbit	Hyperbolic orbit	
t_o = + 0.0000000000 D + 00	TF = + 0.9753400000 D + 03	Check 1 = + 0.1000000000 D + 01
X_1 = – 0.7515331845 D + 03	X_1 = – 0.2063161960 D + 04	Check 2 = + 0.2547716287 D – 20
X_2 = – 0.1719532694 D + 05	X_2 = – 0.9384960103 D + 04	Check 3 = – 0.1671315173 D – 13
X_3 = – 0.1922853605 D + 05	X_3 = – 0.2434255191 D + 05	
$\dot{X}_1$ = – 0.1358441628 D + 01	$\dot{X}_1$ = – 0.1326442061 D + 01	
$\dot{X}_2$ = + 0.7840211728 D + 01	$\dot{X}_2$ = + 0.8144014726 D + 01	
$\dot{X}_3$ = – 0.5483792681 D + 01	$\dot{X}_3$ = – 0.4987127484 D + 01	

Numerical Results

The equations of the present and previous sections are programmed and applied with fixed step size, fourth-order Runge-Kutta-Gill method. Conditions (3.14), (3.15) and (3.16) are used for checking the accuracies of numerical integration. Although the program is developed to include up to any number of Earth's zonal harmonic terms J_n, however, the numerical computations are done with terms up to J_{36}. The output of the program was arranged for each case study in the second and third columns of Tables 1, 2 and 3, where CHECK 1, 2, 3 correspond respectively to the conditions (3.14), (3.15) and (3.16).

Efficiency Study

In order to judge the efficiency of the motion prediction algorithm for trajectories of very short flight times, or order 30 minutes (like those cases of the present paper), the accuracy of its final state prediction should be at least of the order of few centimeters. As the first step for such efficiency study, we need the reference final state of the given case. For the three test cases each for the geopotential model with zonal harmonic terms up to J_{36} we produced a reference state determined by reducing the time step size used in the numerical solution of the system.

$$\ddot{X} + \frac{\mu}{r^3} X = -\frac{\partial V}{\partial X}$$

until five decimal places (~ 10^{-2} meter) stabilized in X at TF, where TF is the flight time. Let this reference state be X_{jR} (TF), X_{jE} (TF) are the final values as obtained by the present method, and X_{jC} (TF) are the corresponding final values as obtained by the solution of the classical equations of motion. Then the efficiency of a method with final values X_{jQ} (TF) (say) can be measured by the magnitude of what we call it, the prediction error PE defined as

$$PE = \left\{ \sum_{j=1}^{3} [X_{jQ}(TF) - X_{jR}(TF)]^2 \right\}^{1/2} \times 10^5 \text{ (in centimeters)} \qquad (3.17)$$

such that, the smaller the value of PE, the higher the efficiency of the prediction algorithm will be.

In Tables 4, 5 and 6, the efficiency studies for the test cases are presented. In the second column of each table, the final state of the case is rewritten for the purpose of comparison, while the third and fourth columns of the tables give the value of the prediction error [computed by equation (3.17)] for the motion algorithm of the presented paper and for the solution of classical equations of motion respectively.

TABLE 4. Efficiency study for the test case No. 1.

j	X_{jR} (TF)	X_{jE} (TF)	X_{jC} (TF)
1	0.1097091933 D + 05	0.109791933 D + 05	0.10969562853 D + 05
2	0.1435479581 D + 04	0.1435479581 D + 04	0.14354637985 D + 04
3	0.4304935816 D + 04	0.4304935816 D + 04	0.43048886713 D + 04
(PE)E = 0.0 cm , (PE)c = 135150.9 cm			

TABLE 5. Efficiency study for the test case No. 2.

j	X_{jR} (TF)	X_{jE} (TF)	X_{jC} (TF)
1	– 0.1835330831 D + 05	– 0.1835330831 D + 05	– 0.18353271824 D + 05
2	+ 0.1095613853 D + 05	+ 0.1095613853 D + 05	+ 0.10956124599 D + 05
3	+ 0.6018407546 D + 04	+ 0.6018407546 D + 04	+ 0.60184309983 D + 04
$(PE)_E$ = 0.0 cm , $(PE)_c$ = 4597.1 cm			

TABLE 6. Efficiency study for the test case No. 3.

j	X_{jR} (TF)	X_{jE} (TF)	X_{jC} (TF)
1	– 0.2063161960 D + 04	– 0.2063161960 D + 04	– 0.20631619132 D + 04
2	– 0.9384960103 D + 04	– 0.9384960103 D + 04	– 0.93849603552 D + 04
3	– 0.2434255191 D + 05	– 0.24342551916 D + 05	– 0.24342551705 D + 05
$(PE)_E$ = 0.0 cm , $(PE)_c$ = 195.3 cm			

It is clear from these tables that, the solution obtained from the classical equations of motion is too far from the requirement nowadays of centimeter class solutions of very short flight time trajectories. On the other hand, the algorithm of the present paper is extremely accurate and is within the class of acceptable solution for very short flight time trajectories.

Summary

A motion prediction algorithm using the Eulerian parameters has been developed for the motion in the Earth's gravitational field with axial symmetry. The algorithm is of recursive nature, and moreover could be used for any conic motion whatever the number of zonal harmonic coefficients may be. Numerical results proved the very highly efficiency and flexibility of the developed algorithm for the nowadays requirements of centimeter class solutions of very short flight time trajectories, which are of extreme importance for military purposes.

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المسألة الاستهلالية لحركة الأقمار الصناعية في المجال التجاذبي الأرضي ذي التماثل المحوري بدلالة متغيرات أويلر

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المستـخلص . في هذا البـحث ، تم تكوين خـوارزميـة عـامة تعـاودية ومستقرة للمسألة الاستهلالية بدلالـة متغيرات أويلر البارامترية ؛ وذلك للتنبؤ الحركي للأقمار الصناعيـة في المجال التجاذبي للأرض ذي التـماثل المحوري . اشتمل البحث على بعض التطبيقات العددية للخوارزميات .